A STOCHASTIC VARIATIONAL APPROACH TO THE VISCOUS CAMASSA-HOLM AND LERAY-ALPHA EQUATIONS

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ABSTRACT. We derive the (d-dimensional) periodic incompressible and viscous Camassa-Holm equation as well as the Leray-alpha equations via stochastic variational principles. We discuss the existence of solution for these equations in the space H^1 using the probabilistic characterisation. The underlying Lagrangian flows are diffusion processes living in the group of diffeomorphisms of the torus. We study in detail these diffusions.

Contents

1. Introduction

In the context of fluid dynamics the one dimensional Camassa-Holm equation

$$(1.1) \qquad \frac{\partial}{\partial t}(u - u'') = -3uu' + 2u'u'' + uu'''$$

was introduced in [5] to describe the motion of unidirectional shallow water waves. The Lagrangian approach consists in looking at the integral flows associated to the velocities u, namely the curves $g(t)(x)$ satisfying

$$(1.2) \qquad \frac{\partial}{\partial t}g(t)(x) = u(t, g(t)(x)), \ g(0)(x) = x.$$

The Lagrangian flows $g(t)(\cdot)$ corresponding to the Camassa-Holm equation are geodesics with respect to the right-invariant induced H^1 metric on a group of (Sobolev) homeomorphisms over the circle. This was proved in [25] and [20] and it corresponds to Arnold's ([3]) characterization of the Euler equation, now replacing the L^2 by the right invariant H^1 norm in the Lagrangian.

Geodesics are minima of length and there is, indeed, a variational principle associated to the equations. The Camassa-Holm Lagrangian flows $g(t)$, with $t \in [0, T]$, can be characterized as critical paths for the action functional

$$(1.3) \qquad\qquad S[g] = \frac{1}{2} E \int_0^T ||\dot{g}(t) \circ g^{-1}(t)||^2_{H^1} dt$$

where $\dot{g}$ denotes the derivative in time of g.

These geodesic equations are a special case of Lagrangian systems treated in Geometric Mechanics via variational principles in general Lie groups ([23]).

In [26], and developing infinite-dimensional geometric methods as in [12], the well-posedness of the problem in the space H^s with $s > \frac{3}{2}$ was proved. Also in [8] the existence of solutions in H^3 was shown. Camassa-Holm equation is also studied in higher dimensions in many works: we refer here to [18] and [26].

Considering this equation in the viscous case, namely

$$\frac{\partial}{\partial t}(u - u'') - \nu(u - u'')'' = -3uu' + 2u'u'' + uu'''$$

and, in higher dimensions,

$$\frac{\partial}{\partial t}(u - \Delta u) - \nu\Delta(u - \Delta u)$$
$$= \quad -u \cdot \nabla(u - \Delta u) - \operatorname{div} u(u - \Delta u) + \sum_j \nabla u^j \cdot \Delta u^j - \frac{1}{2}\nabla|u|^2,$$

we lose the geodesic characterization, as we are no longer dealing with a conservative system. Nevertheless we still have a variational model, in a stochastic framework. The aim of this article is to formulate a stochastic variational principle for the viscous incompressible Camassa-Holm equation and prove an existence result of the critical (Lagrangian) stochastic process. The Lagrangian process provides a solution for this equation: in our stochastic variational framework we replace deterministic Lagrangian paths by semimartingales and consider the classical Lagrangian evaluated on the drift of those semimartingales, this drift playing the role of their (mean) time derivative. The critical paths for the action are diffusions whose drift satisfies Camassa-Holm equation.

The viscous Camassa-Holm equations are also known as the Navier-Stokes-alpha equations. In our work we consider α equal to one, for simplicity of notation (this does not have any implication in the results). A very similar model, known as the Leray-alpha equations, was introduced in [6]. We study also these equations (again, with $\alpha = 1$) from the variational point of view. They are obtained when considering the same action functional, but a different class of admissible variations.

The stochastic variational principle was derived in the case of the L^2 metric and for the Navier-Stokes equations, in the two-dimensional torus, in [7] and later generalized to compact Riemannian manifolds in [2]. The same kind of variational principles can be formulated on general Lie groups: this is the content of reference [1] (c.f. also [9]).

Viscous Camassa-Holm equations were introduced in [10], [11] and [16] as a model of fluid turbulence (c.f. also [17]). Several works are devoted to the existence of solutions for this equation in Sobolev spaces and under different boundary conditions via partial differential equation methods. Just to cite some, we mention [27], [22] or [4].

Our study is probabilistic. It may be regarded from the perspective of stochastic control theory or stochastic geometric mechanics.

We study the incompressible viscous Camassa-Holm equation with periodic boundary conditions in the space variable (i.e. in the d-dimensional torus $\mathbb{T}^d$). In the next section we construct some Brownian motions living in the homeomorphisms group and show the existence of stochastic flows which are perturbations of such Brownian motions by a time dependent drift $u \in L^2([0,T]; H^1(\mathbb{T}^d))$. In section 3 we derive a stochastic variational principle for the H^1 metric and in the following section we prove the existence of a weak solution for the periodic incompressible viscous Camassa-Holm equation under some restrictive assumption. For the Leray-alpha model and using a different class of admissible variations this assumption can be removed: this is done in the last paragraph.

2. Stochastic processes on the homeomorphisms group of the torus

2.1. The group of homeomorphisms of the torus.

Let $\mathbb{T}^d$ be the d-dimensional flat torus. We denote by $G^s, s \geq 0$ the (infinite-dimensional) group of maps $g : \mathbb{T}^d \to \mathbb{T}^d$ such that g and g^{-1} belong to the Sobolev space H^s and such that g keeps the volume measure $d\theta$ invariant, namely $(g)_*(d\theta) = d\theta$. When $s > \frac{d}{2} + 1$ Sobolev imbedding theorems imply that the maps of G^s are diffeomorphisms. Also G^s is a topological group for the composition of maps (not quite a Lie group because left composition is not a smooth operation) and it is a smooth manifold (c.f. [12]).

Let us denote by e the identity of the group, $e(\theta) = \theta$. The tangent space at the identity consists of vector fields on M which are H^s regular and have zero divergence. This space is identified with the Lie algebra of the group. On $G^s \equiv G^s(\mathbb{T}^d)$ we consider the H^1 metric: for $X, Y \in T_e G^s$,

$$< X, Y >_{H^1} = \int_{\mathbb{T}^d} (X(\theta).Y(\theta))d\theta + \int_{\mathbb{T}^d} (\nabla X(\theta).\nabla Y(\theta))d\theta.$$

One extends this metric by right invariance to G^s, namely if $X, Y \in T_g(G^s)$, the space of H^s vector fields over g,

$$(< X, Y >_{H^1})_g = < X \circ g^{-1}, Y \circ g^{-1} >_{H^1}.$$

Note that this metric does not necessarily coincide with the one that defines the topology (this is an example of a *weak Riemannian structure*). We refer to [12] for a detailed study of the geometry of diffeomorphism groups.

We want to construct a basis for the Lie algebra of the group G^s endowed with the H^1 metric. Let $\tilde{\mathbb{Z}}^d$ be a subset of $\mathbb{Z}^d$ where we identify $k, l \in \mathbb{Z}^d$ through the equivalence relation $k \simeq l$ iff $k + l = 0$.

Consider, for each $k \neq 0$, an orthonormal basis $\{\epsilon_k^1, ..., \epsilon_k^{d-1}\}$ of the space $E_k = \{x \in \mathbb{R}^d : k.x = 0\}$, where $k.x$ denotes scalar product. For example, when $d = 2$ we take $\epsilon_k = \frac{1}{|k|}(-k_2, k_1)$ when $k = (k_1, k_2)$. For $d > 2$ there is no canonical choice of basis. Write $\epsilon_{-k} = -\epsilon_k$.

Considering the usual identification of vector fields and functions, $u \to u(\theta).\nabla$, a basis of the Lie algebra of H^s divergence free vector fields on the d-dimensional torus can be defined as

$$\{\lambda_k(s)(\epsilon_k^\alpha)\cos(k.\theta), \lambda_k(s)(\epsilon_k^\alpha)\sin(k.\theta)\}_{k \in \tilde{\mathbb{Z}}^d, \alpha = 1,...,d-1}$$

for some normalizing factors $\lambda_k(s)$, together with the canonical vector fields of $\mathbb{R}^d$.

In this work we typically denote by k, l indeces in $\tilde{\mathbb{Z}}^d$, by $\alpha, \beta = 1, ..., d-1$ indices for the basis of of E_k and $i, j = 1, ..., d$ the d-dimensional components of the torus.

2.2. Brownian motions on G^0.

Let $\{x_k^{\alpha,1}(t, \omega), x_k^{\alpha,2}(t, \omega)$ with $k \in \tilde{\mathbb{Z}}^d, k \neq 0, \alpha = 1, ..., d-1, t \geq 0$, be a sequence of real valued independent standard Brownian motions defined on a given filtered probability space $(\Omega, \mathcal{F}, P)$, with increasing filtration $\mathcal{F}_t$, and let $y(t, \omega)$ be a $\mathbb{T}^d$-valued Brownian motion with components which are also independent from the previous real-valued Brownian motions. We shall drop the probability space parameter in the notations.

Define $\alpha_k^2 = (|k|^2 + 1)^{\frac{r}{2}}$ with $r \geq d+1$. Then

$$(2.1) \qquad x(t)(\theta) = \sum_{k \neq 0} \sum_\alpha \frac{1}{\alpha_k}(\epsilon_k^\alpha)[x_k^{\alpha,1}(t)\cos(k.\theta) + x_k^{\alpha,2}(t)\sin(k.\theta)] + y(t)$$

converges uniformly on $[0, T] \times \mathbb{T}^d$.

The canonical horizontal diffusion corresponding to the Lie algebra valued process $x(t)$ is the solution of the following Stratonovich stochastic differential equation with respect to the filtration $\mathcal{F}_t$, $t \in [0, T]$,

$$(2.2) \qquad\qquad\qquad dg(t) = (\circ dx(t))(g(t)), \qquad g(0) = e.$$

More explicitly, for $i = 1, ..., d$,

$$dg_i(t)(\theta) = \sum_{k \neq 0} \sum_\alpha \frac{1}{\alpha_k}(\epsilon_k^\alpha)_i[\cos(k.g(t)(\theta))\circ dx_k^{\alpha,1}(t) + \sin(k.g(t)(\theta))\circ dx_k^{\alpha,2}(t)] + dy_i(t).$$

$$g_i(0)(\theta) = \theta_i$$

Lemma 2.1. *Equation (2.2) can be written in the Itô form as follows:*
(2.3)
$$dg_i(t)(\theta) = \sum_{k \neq 0} \sum_\alpha \frac{1}{\alpha_k}(\epsilon_k^\alpha)_i[\cos(k.g(t)(\theta))dx_k^{\alpha,1}(t) + \sin(k.g(t)(\theta))dx_k^{\alpha,2}(t)] + dy_i(t)$$

$$g_i(0)(\theta) = \theta_i.$$

Proof. We have,

$$d\cos(k.g(t)).dx_k^{\alpha,1}(t)$$
$$= -\sum_i k_i \sin(k.g(t))dg_i(t).dx_k^{\alpha,1}(t)$$
$$= -\sum_{i,\beta,m \neq 0} k_i \sin(k.g(t))\frac{1}{\alpha_m}(\epsilon_m^\beta)_i[\cos(m.g(t)) \circ dx_m^{\beta,1}(t) + \sin(m.g(t)) \circ dx_m^{\beta,2}(t)].dx_k^{\alpha,1}(t)$$
$$= -\sum_i \frac{1}{\alpha_k}k_i(\epsilon_k^\alpha)_i \sin(k.g(t))\cos(k.g(t))dt$$

and

$$d\sin(k.g(t)).dx_k^{\alpha,2}(t)$$

$$= \sum_i k_i \cos(k.g(t))dg_i(t).dx_k^{\alpha,2}(t)$$

$$= \sum_{i,\beta,m\neq 0} k_i \cos(k.g(t))\frac{1}{\alpha_m}(\epsilon_m^\beta)_i[\cos(m.g(t)) \circ dx_m^{\beta,1}(t) + \sin(m.g(t)) \circ dx_m^{\beta,2}(t)].dx_k^{\alpha,2}(t)$$

$$= \sum_i \frac{1}{\alpha_k}k_i(\epsilon_k^\alpha)_i \sin(k.g(t)) \cos(k.g(t))dt$$

The Itô stochastic contraction is therefore equal to zero and the conclusion follows.

$\square$

We prove the existence of the process $g(t)$ in the case where $r = d + 3$. For this we shall need the following lemma.

Lemma 2.2. *Define*

$$V(\theta) = \sum_{k\neq 0} \frac{\sin^2(k \cdot \theta)}{|k|^{d+3}}.$$

Then there exists a constant $C_1 > 0$ such that for all $0 < |\theta| < \frac{\pi}{4}$,

$$V(\theta) \leq C_1|\theta|^2 \log \frac{1}{|\theta|}.$$

Proof. We have,

$$V(\theta) = \sum_{k\neq 0} \frac{\sin^2(k \cdot \theta)}{|k|^{d+3}} \leq C_d \sum_{k\neq 0} \sum_{i=1}^d \frac{\sin^2(k_i \cdot \theta_i)}{|k|^{d+3}}$$

Using the following inequality of arithmetic and geometric means

$$\frac{\frac{k_i^2}{3} + \frac{k_i^2}{3} + \frac{k_i^2}{3} + (k_1^2 + \cdots + \breve{k}_i^2 + \cdots + k_d^2)}{4} \geq \sqrt[4]{(\frac{k_i^2}{3})^3(k_1^2 + \cdots + \breve{k}_i^2 + \cdots + k_d^2)},$$

i.e. $|k|^4 \geq Ck_i^3(k_1^2 + \cdots + \breve{k}_i^2 + \cdots + k_d^2)^{\frac{1}{2}}$, we obtain $|k|^{d+3} = |k|^4|k|^{d-1} \geq Ck_i^3(k_1^2 + \cdots + \breve{k}_i^2 + \cdots + k_d^2)^{\frac{d}{2}}$, where the symbol $\breve{}$ represents omission of the corresponding term.

Since $\sum_{k\neq 0} \frac{1}{(k_1^2 + \cdots + \breve{k}_i^2 + \cdots + k_d^2)^{\frac{d}{2}}}$ converges, we have

$$\begin{aligned}
V(\theta) &\leq C_d \sum_{i=1}^d \sum_{k\neq 0} \frac{\sin^2(k_i \cdot \theta_i)}{|k|^{d+3}} \\
&\leq C_d \sum_{i=1}^d (\sum_{k\neq 0} \frac{\sin^2(k_i \cdot \theta_i)}{k_i^3})(\sum_{k\neq 0} \frac{1}{(k_1^2 + \cdots + \breve{k}_i^2 + \cdots + k_d^2)^{\frac{d}{2}}}) \\
&\leq \widetilde{C}_d \sum_{i=1}^d \sum_{k\neq 0} \frac{\sin^2(k_i \cdot \theta_i)}{k_i^3} \\
&\leq C \sum_{i=1}^d |\theta_i|^2 \log \frac{1}{|\theta_i|}
\end{aligned}$$

where $\widetilde{C}_d = C_d \sum\limits_{k \neq 0} \frac{1}{(k_1^2 + \cdots + \check{k}_i^2 + \cdots + k_d^2)^{\frac{d}{2}}}$, and the last inequality comes from lemma 2.1 in [13].

Noticing the fact that the function $\xi \to \xi \log \frac{1}{\xi}$ is concave over $]0, 1[$,

$$
\begin{aligned}
2\sum_{i=1}^{d} |\theta_i|^2 \log \frac{1}{|\theta_i|} &= \sum_{i=1}^{d} |\theta_i|^2 \log \frac{1}{|\theta_i|^2} \\
&\leq (\sum_{i=1}^{d} |\theta_i|^2) \log \frac{1}{(\sum_{i=1}^{d} |\theta_i|^2)} \\
&= |\theta|^2 \log \frac{1}{|\theta|^2} \\
&= 2|\theta|^2 \log \frac{1}{|\theta|}
\end{aligned}
$$

Finally, we get the result $V(\theta) \leq C_1 |\theta|^2 \log \frac{1}{|\theta|}$. $\qquad\square$

Theorem 2.3. *For $r = d+3$ the solution $g(t)$ of the stochastic differential equation (2.2) exists and is a continuous process with values in the space of the homeomorphism group G^0.*

Proof. The proof follows essentially the arguments in [13]. For each $\theta \in \mathbb{T}^d$ consider $g^n(t)(\theta)$ the solution of the following s.d.e.:
(2.4)
$$
d\gamma^n(t) = \sum_{|k| \leq 2^n, k \neq 0} \frac{1}{\alpha_k} \sum_{\alpha} (\epsilon_k^\alpha)[\cos(k.\gamma^n(t))dx_k^{\alpha,1}(t) + \sin(k.\gamma^n(t))dx_k^{\alpha,2}(t)] + dy(t)
$$

$$
\gamma^n(0) = \theta.
$$

Denote $\eta_i(t) = \frac{\gamma_i^n(t) - \gamma_i^{n+1}(t)}{4}, i = 1, ..., d$; then the Itô's stochastic contraction is

$$
\begin{aligned}
d\eta_i(t) \cdot d\eta_i(t) &= (\frac{1}{4})^2 \{ \sum_{|k|=1}^{2^n} \frac{1}{\alpha_k^2} \sum_{\alpha} [(\epsilon_k^\alpha)_i]^2 [(\cos k.\gamma^{n+1}(t) - \cos k.\gamma^n(t))^2 \\
&\quad + (\sin k.\gamma^{n+1}(t) - \sin k.\gamma^n(t))^2] \\
&\quad + \sum_{|k|=2^n+1}^{2^{n+1}} \frac{1}{\alpha_k^2} \sum_{\alpha} [(\epsilon_k^\alpha)_i]^2 [\cos^2(k.\gamma^{n+1}(t)) + \sin^2(k.\gamma^{n+1}(t))] \} \\
&\leq (\frac{1}{4})^2 [(d-1) \sum_{|k|=1}^{2^n} \frac{1}{\alpha_k^2} 4 \sin^2(k \frac{\gamma^{n+1}(t) - \gamma^n(t)}{2}) + (d-1) \sum_{|k|=2^n+1}^{2^{n+1}} \frac{1}{\alpha_k^2}].
\end{aligned}
$$

Since $[(\epsilon_k^\alpha)_i]^2 \leq |\epsilon_k^\alpha|^2 \leq 1$, the last inequality holds. We have $\alpha_k^2 = (|k|^2 + 1)^{\frac{d+3}{2}} \geq |k|^{d+3}$. From lemma 2.2 we obtain

$$
\sum_{|k|=1}^{2^n} \frac{1}{\alpha_k^2} 4 \sin^2(k. \frac{\gamma^{n+1}(t) - \gamma^n(t)}{2}) \leq C_1 |\eta(t)|^2 \log \frac{1}{|\eta(t)|}
$$

and

$$\sum_{|k|=2^n+1}^{2^{n+1}} \frac{1}{\alpha_k^2} \leq \sum_{|k|=2^n+1}^{2^{n+1}} \frac{1}{|k|^{d+3}} \leq C_2 2^{-n}.$$

where $C_2 = \sum_{k \neq 0} \frac{1}{|k|^{d+2}}$.

Using Itô's formula, for $p \geq 1$, we obtain

$$d\eta_i^{2p}(t) = 2p\eta_i^{2p-1}(t) \cdot d\eta_i(t) + p(2p-1)\eta_i^{2p-2}(t)d\eta_i(t) \cdot d\eta_i(t).$$

It follows that

$$E^{\mathcal{F}_t}(\eta_i^{2p}(t+\varepsilon) - \eta_i^{2p}(t)) \leq C_1 p \int_t^{t+\varepsilon} E^{\mathcal{F}_t}(|\eta(s)|^{2p} \log \frac{1}{|\eta(s)|^{2p}} ds) + K_p 2^{-n}\varepsilon,$$

where $K_p = C_2 p(2p-1)(\frac{\pi}{4})^{2p-2}$.

Denote $\varphi(t) = E(|\eta(t)|^2)$; we have

$$\varphi'(t) = \sum_i \frac{d}{dt} E(\eta_i^2(t)) \quad \leq \quad C_1 E(|\eta(t)|^2 \log \frac{1}{|\eta(t)|^2}) + C_2 2^{-n}$$

$$\leq \quad C(\varphi(t) \log \frac{1}{\varphi(t)} + 2^{-n}),$$

where the last inequality comes from the fact that the function $\xi \to \xi \log \frac{1}{\xi}$ is concave over $]0,1[$. We can apply Lemma 2.3 in [13] to get

$$\psi'(t) \leq C\psi(t) \log \frac{1}{\psi(t)},$$

where $\psi(t) = \varphi(t) + 2^{-n}$. Now lemma 2.2 in [13] gives

$$\varphi(t) \leq \psi(t) \leq 2^{-n\delta(t)}$$

with $\delta(t) = e^{-Ct}$.

Hence there exist a constant $C > 0$ such that,

$$E(|g^n(t)(\theta) - g^{n+1}(t)(\theta)|^2) \leq C2^{-n\delta(t)}.$$

By the martingale maximal inequality,

$$E(\sup_{0 \leq t \leq T} |g^n(t)(\theta) - g^{n+1}(t)(\theta)|^2) \leq C2^{-n\delta(T)}.$$

Using Borel-Cantelli we deduce that

$$g(t)(\theta) = \lim_n g^n(t)(\theta)$$

exists uniformly in $t \in [0,T]$.

Following [13], one can show that $g(t)$ satisfies Eq.(2.2) and that $g(t)$ is the unique solution of this equation.

We consider now the regularity of $g(t)$. Using the same computation as before, we have the following estimate

$$E(\sup_{0 \leq t \leq T} |g^n(t)(\theta) - g^n(t)(\theta')|^2) \leq C|\theta - \theta'|^{2\delta(T)}.$$

From the triangle inequality, we obtain

$$E(|g^n(t)(\theta) - g^{n+1}(t)(\theta')|^2) \leq C[|\theta - \theta'|^{2\delta(t)} + 2^{-n\delta(t)}].$$

Then we apply the Kolmogorov modification theorem, so that almost surely $g_t^n \to g_t$ uniformly over $\mathbb{T}^d$. Since $\delta(t) \to 1$ as $t \to 0$, for any $0 < \delta < 1$, there exists $t_0 > 0$ such that

$$(2.5) \qquad |g(t)(\theta) - g(t)(\theta')| \le C_{\delta,t}|\theta - \theta'|^\delta, \ t \le t_0.$$

Denote $g_{t,x}^n$ the solution of s.d.e. (2.4). Then for $s \le t_0$, we have (see [19])

$$g_{t_0+s,x}^n(\theta) = g_{s,x^{t_0}}^n(g_{t_0,x}^n(\theta)),$$

where $t \to x^{t_0}(t) = x(t + t_0) - x(t_0)$ is again a Brownian motion. For $t > t_0$, using the flow property and letting $n \to +\infty$

$$g_{t_0+s,x}(\theta) = g_{s,x^{t_0}}(g_{t_0,x}(\theta));$$

together with (2.5) and writing $m = [\frac{t}{t_0}]$ the integral part of $\frac{t}{t_0}$, we obtain

$$(2.6) \qquad |g_{t,x}(\theta) - g_{t,x}(\theta')| \le C_{\delta,t}|\theta - \theta'|^{\delta^{m+1}}.$$

We have $m + 1 \le \frac{t}{t_0} + 1 \le 2\frac{t}{t_0}$. Denote $c_0 = \frac{2}{t_0} \log \frac{1}{\delta}$, then we deduce

$$|g_{t,x}(\theta) - g_{t,x}(\theta')| \le C_t|\theta - \theta'|^{e^{-c_0 t}}$$

from (2.6). Using this inequality, we can show that $g(t)(\cdot)$ is α-Hölder continuous $(0 < \alpha < e^{-c_0 t})$.

Following [13] and [24], the flow property can be used to prove that the stochastic process $g(t)$ lives in the space of homeomorphisms on $\mathbb{T}^d$. $\square$

Let us describe the infinitesimal generator of the stochastic process $g(t) : \mathbb{T}^d \to \mathbb{T}^d$ defined on very special functionals, namely functionals defined θ-pointwise.

Definition. Let f be a C^2 function defined on $\mathbb{T}^d$. On a functional $F(g)(\theta) = f(g(\theta)), \theta \in \mathbb{T}^d$, the infinitesimal generator of the process $g(t) : \mathbb{T}^d \to \mathbb{T}^d$ is defined by

$$L(F)(\theta) = \lim_{t \to 0} \frac{1}{t} E((g(t))^* f(\theta) - f(\theta)),$$

where $(g(t))^* f(\theta) = f(g(t)(\theta))$.

Theorem 2.4. *Let $\mathcal{L}$ be the infinitesimal generator of the stochastic process $g(t)$ defined in (2.2). Then there exist strictly positive constants $c_1, ..., c_d$ such that, for $F(g)(\theta) = f(g(\theta))$,*

$$\mathcal{L}(F(g))(\theta) = \sum_{i=1}^d c_i \partial_{ii}^2 f(g(\theta)), \ f \in C^2(\mathbb{T}^d).$$

If $d = 2$ we have $c_1 = c_2 = c$ and the infinitesimal generator reduces to the Laplacian operator multiplied by c.

Proof. Let us denote $A_k^\alpha = (\epsilon_k^\alpha)\cos(k.\theta)$ and $B_k^\alpha = (\epsilon_k^\alpha)\sin(k.\theta)$. Using Itô's formula we have,

$$
\begin{aligned}
df(g(t)) &= \sum_{k\neq 0}\sum_\alpha \frac{1}{\alpha_k}[(A_k^\alpha f)(g(t)) \circ dx_k^{\alpha,1}(t) + (B_k^\alpha f)(g(t)) \circ dx_k^{\alpha,2}(t)] \\
&\quad + \sum_i \partial_i f(g(t)) \circ dy^i(t) \\
&= \sum_{k\neq 0}\sum_\alpha \frac{1}{\alpha_k}[(A_k^\alpha f)(g(t))dx_k^{\alpha,1}(t) + (B_k^\alpha f)(g(t))dx_k^{\alpha,2}(t)] \\
&\quad + \sum_i \partial_i f(g(t))dy^i(t) \\
&\quad + \frac{1}{2}\sum_{k\neq 0}\sum_\alpha \frac{1}{\alpha_k^2}[(A_k^\alpha(A_k^\alpha f))(g(t)) + (B_k^\alpha(B_k^\alpha f))(g(t))]dt \\
&\quad + \frac{1}{2}\sum_i \partial_{ii}^2 f(g(t))dt
\end{aligned}
$$

We compute the Itô stochastic contractions. We have

$$
A_k^\alpha(A_k^\alpha)f(\theta) = \sum_{i,j}(A_k^\alpha)_i\partial_i(A_k^\alpha)_j\partial_j f(\theta) + \sum_{i,j}(A_k^\alpha)_i(\theta)(A_k^\alpha)_j(\theta)\partial_{ij}^2 f(\theta)
$$

and

$$
B_k^\alpha(B_k^\alpha)f(\theta) = \sum_{i,j}(B_k^\alpha)_i\partial_i(B_k^\alpha)_j\partial_j f(\theta) + \sum_{i,j}(B_k^\alpha)_i(\theta)(B_k^\alpha)_j(\theta)\partial_{ij}^2 f(\theta)
$$

Let us consider the terms with second order derivatives of f. When $i \neq j$ they are

$$
\frac{1}{2}\sum_{k\neq 0,\alpha}\frac{1}{\alpha_k^2}\sum_{i\neq j}((\epsilon_k^\alpha)_i)((\epsilon_k^\alpha)_j)[\cos^2(k.g(t)(\theta)) + \sin^2(k.g(t)(\theta))]\partial_{ij}^2 f(g(t)(\theta))dt
$$

$$
= \frac{1}{2}\sum_{k\neq 0}\sum_\alpha\frac{1}{\alpha_k^2}\sum_{i\neq j}((\epsilon_k^\alpha)_i)((\epsilon_k^\alpha)_j)\partial_{ij}^2 f(g(t)(\theta))dt.
$$

As we sum in all k (and -k), this term is zero.

For $i = j$ we have

$$
\frac{1}{2}\sum_{k\neq 0}\sum_\alpha\frac{1}{\alpha_k^2}\sum_i[(\epsilon_k^\alpha)_i]^2[\cos^2(k.g(t)(\theta)) + \sin^2(k.g(t)(\theta))]\partial_{ii}^2 f(g(t)(\theta))dt
$$

$$
= \frac{1}{2}\sum_{k\neq 0}\sum_\alpha\frac{1}{\alpha_k^2}\sum_i[(\epsilon_k^\alpha)_i]^2\partial_{ii}^2 f(g(t)(\theta))dt.
$$

Define

$$
a_i = \frac{1}{2}\sum_{k\neq 0}\sum_\alpha\frac{1}{\alpha_k^2}[(\epsilon_k^\alpha)_i]^2
$$

We now consider the terms involving first derivatives of f. They are equal to

$$\frac{1}{2}\sum_{k\neq 0,\alpha}\frac{1}{\alpha_k^2}\sum_{i\neq j}((\epsilon_k^\alpha)_i)((\epsilon_k^\alpha)_j)[-k_i cos(k.g(t)(\theta))\sin(k.g(t)(\theta))$$

$$+k_i\sin(k.g(t)(\theta))\cos(k.g(t)(\theta))]\partial_j f(g(t)(\theta))dt = 0.$$

Therefore these terms vanish.

We conclude that the infinitesimal generator is given by $\mathcal{L}(F(g))(\theta) = \sum_{i=1}^{d} c_i\partial_{ii}^2 f(g(\theta))$, $f \in C^2(\mathbb{T}^d)$, with $c_i = a_i + \frac{1}{2}$.

When $d = 2$ we have only one α, $(\epsilon_k)_1 = -\frac{k_2}{|k|^2}$ and $(\epsilon_k)_2 = \frac{k_1}{|k|^2}$ (for $k = (k_1, k_2)$), from which the result follows, in this case with $c = \frac{1}{2}(\sum_{k\neq 0}\frac{k_1^2}{\alpha_k^2} + 1)$.   □

Remark 2.5.

1) We can consider only $x(t) = y(t)$. In this case the stochastic process is the standard Brownian motion and the generator of the process the Laplacian.

2) We can obtain the same result (up to a modification of the constants in front of the second order differential operator) by considering a Brownian motion defined by a finite sum, namely

$$x(t)(\theta) = \sum_{|k|\leq N, k\neq 0}\sum_\alpha \frac{1}{\alpha_k}(\epsilon_k^\alpha)[x_k^{\alpha,1}(t)\cos(k.\theta) + x_k^{\alpha,2}(t)\sin(k.\theta)] + y(t)$$

Then the corresponding Lagrangian flows are diffeomorphisms by well known results (c.f. [21]).

Let us consider a time-dependent vector field $u : [0, T] \to H^1(\mathbb{T}^d)$, $u \in L^2([0, T]; H^1)$, with $\operatorname{div} u(t, \cdot) = 0$ for all t. We can associate to $u(t)$ the following $\mathcal{F}_t$ stochastic differential equation:

$$dg_i^u(t)(\theta) = u_i(t, g^u(t))dt + \sum_{k\neq 0,\alpha}\frac{1}{\alpha_k}(\epsilon_k^\alpha)_i[\cos(k.g^u(t)(\theta))\sqrt{\frac{\nu}{c_i}}\circ dx_k^{\alpha,1}(t)$$

$$(2.7)\qquad\qquad + \sin(k.g^u(t)(\theta))\sqrt{\frac{\nu}{c_i}}\circ dx_k^{\alpha,2}(t)] + dy(t),$$

with $g_i^u(0)(\theta) = \theta_i$, where c_i denote the constants defined in Theorem 2.4., $t \in [0, T]$. When $c_i = c \ \forall i$ it reads,

$$dg^u(t) = (u(t)dt + \sqrt{\frac{\nu}{c}}\circ dx(t))(g^u(t)), \ g^u(0) = e.$$

This equation can be also written with the Stratonovich differentials replaced by Itô ones, as the Itô contraction vanishes (lemma 2.1).

The rest of this section is devoted to show that this equation defines a stochastic flow.

Theorem 2.6. *Let u belong to the space $L^2([0, T]; H^1(\mathbb{T}^d))$ with $\operatorname{div} u(t, \cdot) = 0$ for all t. Then there exists a stochastic flow g^u which is a solution of the stochastic differential equation (2.7).*

Proof. This result can be found in [15]. There the authors assume that the time dependent vector field takes values in $L^2([0,T]; H^q(\mathbb{T}^d))$ for $q > 2$ but their proof still holds for $q = 2$. They actually prove the existence of a strong solution (c.f. also [7] for a weak solution). Concerning the flow property, the inverse of g_t^u is well defined and satisfies the sde

$$dX_i(t)(\theta) = -u_i(t, X(t))dt + \sum_{k \neq 0,\alpha} \frac{1}{\alpha_k} (\epsilon_k^\alpha)_i [\cos(k.X(t)(\theta)) \sqrt{\frac{\nu}{c_i}} \circ d\hat{x}_k^{\alpha,1}(t)$$

$$+ \sin(k.X(t)(\theta)) \sqrt{\frac{\nu}{c_i}} \circ d\hat{x}_k^{\alpha,2}(t)]$$

where $\hat{x}_k^{\alpha,i}(t) = x_k^{\alpha,i}(T) - x_k^{\alpha,i}(T-t)$ are time-reversed Brownian motions.

$\square$

Corollary 2.7. *Suppose that u satisfies the hypothesis of Theorem 2.6. and let $g^u(t)$ be the solution of Eq.(2.7). Then the infinitesimal generator of this process, when computed at functionals of the form $F(g)(\theta) = f(g(\theta))$, is given by*

$$\mathcal{L}^u(F(g))(\theta) = u(t, g(\theta)).\nabla f(g(\theta)) + \sum_{i=1}^d c_i \partial_{ii}^2 f(g(\theta)), \ \forall f \in C^2(\mathbb{T}^d).$$

3. STOCHASTIC VARIATIONAL PRINCIPLE FOR THE CAMASSA-HOLM EQUATION

Let $\mathcal{S}$ denote the set of continuous semimartingales taking values in the measure-preserving homeomorphism group of $\mathbb{T}^d$.

We consider D_t the generalized time derivative of semimartingales. If F is a smooth function on $\mathbb{T}^d$ and $\xi \in \mathcal{S}$,

$$D_t F(\xi_t) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} [E_t F(\xi(t+\epsilon)) - F(\xi(t))],$$

where E_t denotes conditional expectation with respect to $\mathcal{F}_t$. When $\xi = g^u \in \mathcal{S}_0$, we have

$$D_t g_t^u = u(t, g_t^u).$$

For functionals defined on $\mathcal{S}$ we consider the following variations: to a vector field $v \in C^1([0,T] \times C^\infty(\mathbb{T}^d))$ with $v(0, \cdot)v(T, \cdot) = 0$ and $\operatorname{div} v(t, \cdot) = 0$ for all t, we associate $e_t(v)$ the solution of the ordinary differential equation

$$\frac{d}{dt} e_t^v(\theta) = \dot{v}(t, e_t^v(\theta)), \ e_0^v(\theta) = \theta.$$

The admissible variations of $\xi \in \mathcal{S}$ will be the set of $e_t^v(\xi_t)$, (which are still semimartingales in $\mathcal{S}$).

Definition. Let J be a real-valued functional defined on $\mathcal{S}$. Consider left and right derivatives of J at a semimartingale ξ along directions e^v, $v \in C^1([0,T] \times C^\infty(\mathbb{T}^d))$ with $v(0, \cdot) = v(T, \cdot) = 0$, namely

$$(D_l)_{e^v_\cdot} J[\xi] = \frac{d}{d\epsilon}\Big|_{\epsilon=0} J[e^{\epsilon v}_\cdot \circ \xi(\cdot)]$$

$$(D_r)_{e_\cdot^v} J[\xi] = \frac{d}{d\epsilon}\Big|_{\epsilon=0} J[\xi(\cdot) \circ e_\cdot^{\epsilon v}]$$

A semimartingale ξ is said to be critical for J if

$$(D_l)_{e_\cdot^v} J[\xi] = (D_r)_{e_\cdot^v} J[\xi] = 0$$

for all v as above.

We can now formulate our stochastic variational principle. If g_t^u is a solution of (2.7) for some $u \in L^2([0,T]; H^1)$, div $u(t,\cdot) = 0$, then its drift is of the form $D_t g_t^u = u(t, g_t^u)$. For such a process define the following stochastic action:

$$A[g^u] = \frac{1}{2}E \int_0^T ||(D_t g_t^u) \circ (g_t^u)^{-1}(\cdot)||_{H^1}^2 dt$$

$$(3.2) \qquad\qquad = \frac{1}{2}\int_0^T ||u(t,\cdot)||_{H^1}^2 dt$$

We have the following

Theorem 3.1. *Let g_t^u be solution of (2.7) for some $u \in L^2([0,T]; H^1(\mathbb{T}^d))$ with div $u(t,\cdot) = 0$ for all $t \in [0,T]$. Then g^u is critical for the action functional A if and only if there exists $p \in L^2([0,T]; H^1)$ such that the vector field $u(t)$ satisfies in the weak (L^2) sense the viscous Camassa-Holm equation*

$$\frac{\partial}{\partial t}(u - \mathcal{L}u) - \nu\mathcal{L}(u - \mathcal{L}u) = -u \cdot \nabla(u - \mathcal{L}u) + \sum_j \nabla u^j \cdot \mathcal{L}u^j - \nabla p$$

with $u(T,\cdot) = u_T(\cdot)$, div $u(t,\cdot) = 0 \; \forall t$.

When $\mathcal{L} = \Delta$, it is the usual Camassa-Holm equation. Since the operator $\mathcal{L}$ has similar properties to the standard Laplacian operator Δ, the proof for $\mathcal{L}$ is analogous and we shall write it for Δ.

Recall also that $\mathcal{L}$ reduces to the Laplacian when $d = 2$ and that, by choosing $x(t) = y(t)$ we can always consider the Laplacian case for $d > 2$.

Proof of Theorem 3.1. As the metric is right-invariant we only need to consider left derivatives. Let $\varepsilon > 0$. Since $e_0^v(\theta) = \theta$, we have

$$e_t^{\varepsilon v} = e + \varepsilon \int_0^t \dot{v}(s, e_s^{\varepsilon v}) ds$$

and

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} e_t^{\varepsilon v} = \int_0^t \dot{v}(s, \theta) ds = v(t, \theta).$$

We denote $e_\varepsilon(t, \theta) = e_t^{\varepsilon v}(\theta)$ and $g_t^u = g_t$. We have,

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} A[e_\varepsilon \circ g] = \sum_i E \int\int < \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\{D[e_\varepsilon^i(t, g_t)] \circ (e_\varepsilon(g_t))^{-1}\}, u^i(t, \theta) > dt d\theta$$

$$+ \sum_{i,j} E \int\int < \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\partial_j\{D[e_\varepsilon^i(t, g_t)] \circ (e_\varepsilon(g_t))^{-1}\}, \partial_j u^i(t, \theta) > dt d\theta$$

$$\triangleq A_1 + A_2$$

Since

$$D[e_\varepsilon^i(t, g_t)] = \partial_t e_\varepsilon^i(t, g_t) + (u \cdot \nabla e_\varepsilon^i)(t, g_t) + \nu \Delta e_\varepsilon^i(t, g_t)$$

and

$$g_t \circ (e_\varepsilon(g_t))^{-1} = g_t \circ g_t^{-1} \circ e_\varepsilon^{-1} = e_\varepsilon^{-1},$$

we have

$$\frac{d}{d\varepsilon}|_{\varepsilon=0}\{D[e_\varepsilon^i(t, g_t)] \circ (e_\varepsilon(g_t))^{-1}\}$$

$$= \frac{d}{d\varepsilon}|_{\varepsilon=0}\{\partial_t e_\varepsilon^i(t, e_\varepsilon^{-1}) + \sum_l u^l(t, e_\varepsilon^{-1})\partial_l e_\varepsilon^i(t, e_\varepsilon^{-1}) + \nu \Delta e_\varepsilon^i(t, e_\varepsilon^{-1})\}$$

$$= \partial_t v^i(t, \theta) + \sum_l [u^l(t, \theta)\partial_l v^i(t, \theta) - \partial_l u^i(t, \theta)v^l(t, \theta)] + \nu \Delta v^i(t, \theta)$$

and

$$\frac{d}{d\varepsilon}|_{\varepsilon=0}\partial_j\{D[e_\varepsilon^i(t, g_t)] \circ (e_\varepsilon(g_t))^{-1}\}$$

$$= \partial_j \frac{d}{d\varepsilon}|_{\varepsilon=0}\{D[e_\varepsilon^i(t, g_t)] \circ (e_\varepsilon(g_t))^{-1}\}$$

$$= \partial_j\{\partial_t v^i(t, \theta) + \sum_l [u^l(t, \theta)\partial_l v^i(t, \theta) - \partial_l u^i(t, \theta)v^l(t, \theta)] + \nu \Delta v^i(t, \theta)\}$$

$$= \partial_t \partial_j v^i + \sum_l [\partial_j u^l \partial_l v^i + u^l \partial_{jl}^2 v^i - \partial_{jl}^2 u^i v^l - \partial_l u^i \partial_j v^l] + \nu \Delta \partial_j v^i.$$

Then

$$A_1 = \sum_i E \int \int < \frac{d}{d\varepsilon}|_{\varepsilon=0}\{D[e_\varepsilon^i(t, g_t)] \circ (e_\varepsilon(g_t))^{-1}\}, u^i(t, \theta) > dt d\theta$$

$$= \sum_i E \int \int < \partial_t v^i(t, \theta) + \sum_l [u^l(t, \theta)\partial_l v^i(t, \theta) - \partial_l u^i(t, \theta)v^l(t, \theta)] + \nu \Delta v^i(t, \theta), u^i(t, \theta) > dt d\theta$$

$$= \sum_i [-\int \int \partial_t u^i v^i dt d\theta + \nu \int \int \Delta u^i v^i dt d\theta - \int \int (\text{div } u)u^i v^i dt d\theta$$

$$- \int \int (u \cdot \nabla)u^i v^i dt d\theta - \frac{1}{2} \int \int \partial_i |u|^2 v^i dt d\theta]$$

$$= \int \int < \partial_t u, v > dt d\theta + \int \int < \nu \Delta u, v > dt d\theta - \int \int < (\text{div } u)u, v > dt d\theta$$

$$- \int \int < (u \cdot \nabla)u, v > dt d\theta - \int \int < \frac{1}{2}\nabla |u|^2, v > dt d\theta,$$

$$
\begin{aligned}
A_2 &= \sum_{i,j} E \int \int < \frac{d}{d\varepsilon}|_{\varepsilon=0}\partial_j\{D[e^i_\varepsilon(t,g_t)] \circ (e_\varepsilon(g_t))^{-1}\}, \partial_j u^i > dt d\theta \\
&= \sum_{i,j} E \int \int < \partial_t\partial_j v^i + \sum_l [\partial_j u^l \partial_l v^i + u^l \partial^2_{jl}v^i - \partial^2_{jl}u^i v^l - \partial_l u^i \partial_j v^l] + \nu\Delta\partial_j v^i, \partial_j u^i > dt d\theta \\
&= \sum_{i,j} [\int \int \partial_t\Delta u^i v^i dt d\theta - \nu \int \int \Delta^2 u^i v^i dt d\theta + \int \int \operatorname{div} u \ \Delta u^i v^i dt d\theta \\
&\quad + \int \int (u \cdot \nabla)\Delta u^i v^i dt d\theta + \int \int \partial_i u^j \Delta u^j v^i dt d\theta] \\
&= \int \int < \partial_t\Delta u, v > dt d\theta - \int \int < \nu\Delta^2 u, v > dt d\theta + \int \int < \operatorname{div} u \ \Delta u, v > dt d\theta \\
&\quad + \int \int < (u \cdot \nabla)\Delta u, v > dt d\theta + \sum_j \int \int < \nabla u^j \Delta u^j, v > dt d\theta,
\end{aligned}
$$

Therefore, using the condition $\operatorname{div} u = 0$,

$$
\frac{d}{d\varepsilon}|_{\varepsilon=0} A[e_\varepsilon \circ g^u] = 0
$$

is equivalent to the equation

$$
\frac{\partial}{\partial t}(u - \Delta u) - \nu\Delta(u - \Delta u)
$$
$$
= -u \cdot \nabla(u - \Delta u) + \sum_j \nabla u^j \cdot \Delta u^j - \nabla p,
$$

satisfied in the weak sense (since v is arbitrary).

Remark 3.2.

The proof of Theorem 3.1 can be also found, in a more general group-theoretical framework, in [1], where it was written for the case of the two-dimensional torus.

4. EXISTENCE OF A CRITICAL DIFFUSION

In this paragraph we discuss the existence of a critical diffusion, whose drift, a posteriori, will be a H^1 solution of the Camassa-Holm equation.

Recall that $\mathcal{S}$ is the set of continuous semimartingales taking values in the measure-preserving homeomorphism group of the torus. $\mathcal{S}_0$ will denote the subset of $\mathcal{S}$ consisting of diffusions g^u that verify equation Eq.(2.7) for some drift $u \in L^2([0,T]; H^1(\mathbb{T}^d))$ with $\operatorname{div} u(t, \cdot) = 0$ for all t. Let us consider the set of semimartingales of the form $g(t) = \eta(t, g^u_t(\theta))$, with $g^u \in \mathcal{S}_0$, η smooth in both variables and a measure-preserving diffeomorphism in θ with η_t^{-1} also smooth. Denote this set by $\mathcal{S}_1$. We have $\mathcal{S}_0 \subset \mathcal{S}_1 \subset \mathcal{S}$.

Notice that, if g^u_t is a semimartingale in $\mathcal{S}_0$ then all variations $e^v_t(g^u_t)$, v as above, belong to $\mathcal{S}_1$ and have a drift in $L^2([0,T]; H^1)$.

Fix a vector field $z \in L^2([0,T]; L^2)$ and a constant $c > 0$. The action functional, defined on the set of semimartingales in $\mathcal{S}_1$ for which the corresponding drifts satisfy the condition

$$
\int \int < u(t,\theta), z(t,\theta > dt d\ \theta \geq c,
$$

is bounded below. Let α be its infimum. Suppose that this infimum belongs to $\mathcal{S}_0$. We consider $g^m(t)$ a minimizing sequence. If the minimum is attained in $\mathcal{S}_0$ we can assume that $g^m(t) = g^{u_m}(t)$, with $u_m \in L^2([0,T]; H^1)$.

We have the convergence $A[g^m(\cdot)] \to \alpha$ as $n \to \infty$. The sequence $A[g^m(\cdot)] = \|u_m\|^2_{L^2([0,T];H^1)}$ is bounded, therefore there exists a subsequence u_{m_j} of u_m that converges with respect to the weak topology, more precisely there exists $u \in L^2([0,T]; H^1)$ such that

$$u_{m_j} \to u, \text{ weakly in } L^2([0,T]; H^1).$$

The limit function u satisfies the assumptions of Theorem 2.6. Then we can construct a stochastic process $g^u(t)$ in $\mathcal{S}_0$ as solution of the stochastic differential equation (2.7). Since the norm is weakly lower semi-continuous, we have

$$A[g^u(\cdot)] \leq \lim_{j \to \infty} A[g^{m_j}(\cdot)],$$

we deduce that $A[g^u(\cdot)] = \alpha$ and $g^u(t)$ is a minimum.

The curve of vector fields $u(t, \cdot)$ satisfies the incompressible viscous Camassa-Holm equation and it satisfies the condition

$$\int \int <u(t,\theta), z(t,\theta> dt d\theta \geq c,$$

which is preserved by weak limits.

We have therefore proved the following result,

Theorem 4.1. *Let $z \in L^2([0,T]; L^2)$ be a vector field and c a positive constant. There exists a semimartingale $g(t)$ in the class $\mathcal{S}_1$ which realizes the minimum of the action functional A. If the minimum belongs to $\mathcal{S}_0$ then the corresponding drift $u(t, \cdot)$ satisfies the incompressible viscous Camassa-Holm equation in the weak L^2 sense. Moreover $\int \int <u(t,\theta), z(t,\theta> dt d\theta \geq c$.*

5. The Leray-alpha equations

The assumption that the minimum of the action belongs to the space $\mathcal{S}_0$ considered in last paragraph, is, of course, a priori quite strong. In this section we consider the Leray-alpha equations (with $\alpha = 1$) and we work with different variations that preserve the class of semimartingales $\mathcal{S}_0$. This allows us to remove this assumption and obtain a stronger result for this model.

For simplicity we assume in this section that $c_i = c$ $\forall i$.

The incompressible Leray-alpha equation, with $\alpha = 1$, is

$$\frac{\partial}{\partial t}(u - \Delta u) - \nu \Delta(u - \Delta u) = -u \cdot \nabla(u - \Delta u) - \nabla p$$

with $\operatorname{div} u(t, \cdot) = 0$ for all $t \in [0, T]$.

We consider the same action functional as before, namely

$$A(g^u) = \frac{1}{2} E \int_0^T \|(D_t g_t^u) \circ (g_t^u)^{-1}(\cdot)\|^2_{H^1} dt.$$

The admissible variations to be of the form g_t^ε satisfying the following equation

$$(5.1) \qquad dg_t^\varepsilon = (\sqrt{\frac{\nu}{c}} \nabla e_t^{\varepsilon v} dx(t))(g_t^\varepsilon) + [\partial_t e_t^{\varepsilon v} + (u \cdot \nabla) e_t^{\varepsilon v} + \nu \Delta e_t^{\varepsilon v}](g_t^\varepsilon) dt,$$

In particular we work only in the class $\mathcal{S}_0$. Considering the notion of criticality with respect to this new class of admissible variations, the following result holds

Theorem 5.1. *Let g_t^u be solution of (2.7) for some $u \in L^2([0,T]; H^1(\mathbb{T}^d))$ with* $\operatorname{div} u(t, \cdot) = 0$ *for all $t \in [0,T]$. Then g^u is critical for the action functional A and with respect to the admissible class of variations (5.1) if and only if there exists $p \in L^2([0,T]; H^1)$ such that the vector field $u(t)$ satisfies in the weak (L^2) sense the equation*

$$\frac{\partial}{\partial t}(u - \Delta u) - \nu \Delta(u - \Delta u) = -u \cdot \nabla(u - \Delta u) - \nabla p$$

with $\operatorname{div} u(t, \cdot) = 0 \ \forall t$.

Proof. The proof is similar to the one of Theorem 3.1. We have

$$
\begin{aligned}
\frac{d}{d\varepsilon}|_{\varepsilon=0} A[g^\varepsilon] &= \sum_i E \int \int < \frac{d}{d\varepsilon}|_{\varepsilon=0}\{D(g_t^\varepsilon)^i \circ (g_t^\varepsilon)^{-1}\}, u^i(t,\theta) > dt d\theta \\
&\quad + \sum_{i,j} E \int \int < \frac{d}{d\varepsilon}|_{\varepsilon=0}\partial_j\{D(g_t^\varepsilon)^i \circ (g_t^\varepsilon)^{-1}\}, \partial_j u^i(t,\theta) > dt d\theta \\
&\triangleq A_1 + A_2.
\end{aligned}
$$

Since

$$D(g_t^\varepsilon)^i = [\partial_t(e_t^{\varepsilon v})^i + (u \cdot \nabla)(e_t^{\varepsilon v})^i + \nu \Delta(e_t^{\varepsilon v})^i](g_t^\varepsilon)$$

and

$$D(g_t^\varepsilon)^i \circ (g_t^\varepsilon)^{-1} = \partial_t(e_t^{\varepsilon v})^i + (u \cdot \nabla)(e_t^{\varepsilon v})^i + \nu \Delta(e_t^{\varepsilon v})^i,$$

we have

$$
\begin{aligned}
&\frac{d}{d\varepsilon}|_{\varepsilon=0}\{D(g_t^\varepsilon)^i \circ (g_t^\varepsilon)^{-1}\} \\
&= \frac{d}{d\varepsilon}|_{\varepsilon=0}\{\partial_t(e_t^{\varepsilon v})^i + (u \cdot \nabla)(e_t^{\varepsilon v})^i + \nu \Delta(e_t^{\varepsilon v})^i\} \\
&= \partial_t v^i(t,\theta) + \sum_l u^l(t,\theta)\partial_l v^i(t,\theta) + \nu \Delta v^i(t,\theta)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{d}{d\varepsilon}|_{\varepsilon=0}\partial_j\{D(g_t^\varepsilon)^i \circ (g_t^\varepsilon)^{-1}\} \\
&= \partial_j \frac{d}{d\varepsilon}|_{\varepsilon=0}\{D(g_t^\varepsilon)^i \circ (g_t^\varepsilon)^{-1}\} \\
&= \partial_j\{\partial_t v^i(t,\theta) + \sum_l u^l(t,\theta)\partial_l v^i(t,\theta) + \nu \Delta v^i(t,\theta)\} \\
&= \partial_t \partial_j v^i + \partial_j[\sum_l u^l(t,\theta)\partial_l v^i(t,\theta)] + \nu \Delta \partial_j v^i.
\end{aligned}
$$

Then

$$A_1 = \sum_i E \int\int < \frac{d}{d\varepsilon}|_{\varepsilon=0}\{D(g_t^\varepsilon)^i \circ (g_t^\varepsilon)^{-1}\}, u^i(t,\theta) > dtd\theta$$

$$= \sum_i E \int\int < \partial_t v^i(t,\theta) + \sum_l u^l(t,\theta)\partial_l v^i(t,\theta) + \nu\Delta v^i(t,\theta), u^i(t,\theta) > dtd\theta$$

$$= \sum_i [-\int\int \partial_t u^i v^i dtd\theta + \nu\int\int \Delta u^i v^i dtd\theta - \int\int (u\cdot\nabla)u^i v^i dtd\theta]$$

$$= -\int\int < \partial_t u, v > dtd\theta + \int\int < \nu\Delta u, v > dtd\theta - \int\int < (u\cdot\nabla)u, v > dtd\theta,$$

$$A_2 = \sum_{i,j} E \int\int < \frac{d}{d\varepsilon}|_{\varepsilon=0}\partial_j\{D(g_t^\varepsilon)^i \circ (g_t^\varepsilon)^{-1}\}, \partial_j u^i > dtd\theta$$

$$= \sum_{i,j} E \int\int < \partial_t\partial_j v^i + \partial_j[\sum_l u^l(t,\theta)\partial_l v^i(t,\theta)] + \nu\Delta\partial_j v^i, \partial_j u^i > dtd\theta$$

$$= \sum_{i,j}[\int\int \partial_t\Delta u^i v^i dtd\theta - \nu\int\int \Delta^2 u^i v^i dtd\theta - \int\int \sum_l u^l(t,\theta)\partial_l v^i(t,\theta)\Delta u^i(t,\theta)dtd\theta]$$

$$= \sum_{i,j}[\int\int \partial_t\Delta u^i v^i dtd\theta - \nu\int\int \Delta^2 u^i v^i dtd\theta + \int\int \sum_l v^i(t,\theta)u^l(t,\theta)\partial_l\Delta u^i(t,\theta)dtd\theta]$$

$$= \int\int < \partial_t\Delta u, v > dtd\theta - \int\int < \nu\Delta^2 u, v > dtd\theta + \int\int < (u\cdot\nabla)\Delta u, v > dtd\theta,$$

So $\frac{d}{d\varepsilon}|_{\varepsilon=0}A[g^\varepsilon] = 0$ is equivalent to

$$\partial_t(u-\Delta u) - \nu\Delta(u-\Delta u) = -u\cdot\nabla(u-\Delta u) - \nabla p$$

with $\operatorname{div} u(t,\cdot) = 0$ for all $t \in [0,T]$. The equation holds in the weak sense. $\qquad\square$

As the admissible variations preserve the class $\mathcal{S}_0$, one can show by the same methods used in Theorem 4.1, the following

Theorem 5.2. *Let $z \in L^2([0,T]; L^2)$ be a vector field and c a positive constant. There exists a semimartingale $g(t)$ in the class $\mathcal{S}_0$ which realizes the minimum of the action functional A. Then the corresponding drift $u(t,\cdot)$ satisfies the incompressible Leray-alpha equation equation in the weak L^2 sense. Moreover $\int\int < u(t,\theta), z(t,\theta > dtd\theta \geq c$.*

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